Asymptotic Analysis for Magnetic Resonance Elastography

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Let $\Omega \subset \mathbb{R}^3$ which is a bounded domain with Lipschitz continuous boundary $\partial\Omega$ be the reference domain we considered (i.e. a part of the living body). Based on the setup of MRE experiment done by M. Suga in Chiba University, Japan, we have to assume that the input has to be time harmonic and it is given on a part Γ_D of surface $\partial\Omega$ of the living body while the rest of the part Γ_N is traction free. Here $\Gamma_D, \Gamma_N \subset \partial\Omega$ are open sets such that $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, $\Gamma_D \neq \emptyset$, $\Gamma_D \cap \Gamma_N = \emptyset$ and $\partial\Gamma_D, \partial\Gamma_N$ are of piecewise $C^{1,1}$ class. For simplicity, we assume that our living body of soft tissues is isotropic viscoelastic, then the wave displacement field $\mathbf{U}(t,x)$ in the medium can be described by the following mixed problem:

$$\begin{cases} \rho\partial_t^2\mathbf{U} = \Big\{\nabla[\lambda\nabla\cdot\mathbf{U}] + \nabla\cdot[2\mu\varepsilon(\mathbf{U})]\Big\} + \Big\{\nabla[\zeta\nabla\cdot\partial_t\mathbf{U}] + \nabla\cdot[2\eta\varepsilon(\partial_t\mathbf{U})]\Big\} & \text{in } (0,+\infty)\times\Omega, \\ \mathbf{U}(t,x) = \chi(t)e^{i\omega t}\mathbf{f}(x) & \text{on } (0,+\infty)\times\Gamma_D, \\ \partial_{\boldsymbol{\nu}}\mathbf{U}(t,x) := \Big\{[\lambda(\nabla\cdot\mathbf{U}) + 2\mu\varepsilon(\mathbf{U})] + [\zeta(\nabla\cdot\partial_t\mathbf{U}) + 2\eta\varepsilon(\partial_t\mathbf{U})]\Big\}\boldsymbol{\nu} = 0 & \text{on } (0,+\infty)\times\Gamma_N, \\ \mathbf{U} = \partial_t\mathbf{U} = 0 & \text{on } \{0\}\times\Omega, \end{cases}$$

Here ω is a given angular motion frequency (LF, ~50-1000 Hz), $\chi(t) \in C^\infty([0,\infty))$, $(\chi(t) = 0\,(0 \le t \le 1/2),\ 1\,(t \ge 1)$ is a cutoff function and $\boldsymbol{\nu}$ is the outward unit normal to $\partial\Omega$. $\lambda(x)$ and $\mu(x)$ are the Lamé modulus, while $\zeta(x)$ and $\eta(x)$ are the viscosity coefficients. Especially, $\mu(x)$ and $\eta(x)$ are called shear modulus and shear viscosity respectively. Physically well known Possion's ratio ν is given by $\nu = \lambda/2(\lambda+\mu)$. Moreover we have the strong convexity:

$$\mu > \delta, \quad \eta > \delta, \quad 3\lambda + 2\mu > \delta, \quad 3\zeta + 2\eta > \delta \quad (\text{a.e. in } \Omega)$$

for some constant $\delta > 0$.

We can prove the exponential decay property in time of this dynamical wave displacement field $\mathbf{U}(t,x)$, and then the governing model becomes the following mixed boundary value problem:

$$\begin{cases} [\nabla(\lambda\nabla\cdot\mathbf{u}) + \nabla\cdot(2\mu\varepsilon(\mathbf{u})) + \rho\omega^2\mathbf{u}] + i\omega[\nabla(\zeta\nabla\cdot\mathbf{u}) + \nabla\cdot(2\eta\varepsilon(\mathbf{u}))] = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{f} & \text{on } \Gamma_D, \\ \partial_{\boldsymbol{\nu}}\mathbf{u} := [\lambda\nabla\cdot\mathbf{u} + 2\mu\varepsilon(\mathbf{u})]\boldsymbol{\nu} + i\omega[\zeta\nabla\cdot\mathbf{u} + 2\eta\varepsilon(\mathbf{u})]\boldsymbol{\nu} = 0 & \text{on } \Gamma_N. \end{cases} \tag{1}$$

In a living body of soft tissues, usually the Poisson's ratio is near 0.5 which means λ is much more larger that μ. A typical order of value for λ is GPa while μ is of the order of kPa which is the case for nearly incompressible material. Although, we can use the model (1), a very large λ and too many viscoelasticity coefficients may cause noise in MRE measurement and make the inversion analysis complex. We still desire to find an approximate model which doesn't contain any large λ.

By letting $\lambda(x) = \alpha\widetilde{\lambda}(x)$, $\mu(x) = \beta\widetilde{\mu}(x)$, $\kappa = \alpha/\beta$, $|\kappa| \gg 1$, $\widetilde{\zeta} := \beta^{-1}\zeta$, $\widetilde{\eta} := \beta^{-1}\eta$, $\widetilde{\rho} := \beta^{-1}\rho$ with some constants α, β such that $|\widetilde{\lambda}|/|\widetilde{\mu}| = O(1)$ and $p := -\beta^{-1}\lambda\nabla\cdot\mathbf{u} = -\kappa\widetilde{\lambda}\nabla\cdot\mathbf{u}$. We apply the asymptotic analysis, by representing $(\mathbf{u},\ p)$ in the formal series: $\mathbf{u} = \sum_{j=0}^{+\infty}\kappa^{-j}\mathbf{u}_{-j}$, $p = \sum_{j=0}^{+\infty}\kappa^{-j}p_{-j}$ and a simple computation, we have for $(\mathbf{u}_0,\ p_0)$:

$$\begin{cases} \nabla\cdot\Big[2(\widetilde{\mu} + i\omega\widetilde{\eta})\varepsilon(\mathbf{u}_0)\Big] - \nabla p_0 + \widetilde{\rho}\omega^2\mathbf{u}_0 = 0 & \text{in } \Omega, \\ \nabla\cdot\mathbf{u}_0 = 0 & \text{in } \Omega, \\ \mathbf{u}_0 = \mathbf{f} & \text{on } \Gamma_D, \\ \partial_{\boldsymbol{\nu}}\mathbf{u}_0 := \Big[2(\widetilde{\mu} + i\omega\widetilde{\eta})\varepsilon(\mathbf{u}_0) - p_0\Big]\boldsymbol{\nu} = 0 & \text{on } \Gamma_N. \end{cases}$$

Remark. $\|\mathbf{u} - \mathbf{u}_0\| = O(\kappa^{-1})$.